# PROJECTIVE MATRIX TRANSFORMATIONS IN MICROWAVE NETWORK THEORY

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### Abstract

Recent theoretical investigations reveal the dominant role played by a new type of matrix transformation in the theory of microwave networks composed of multiport elements; this is an extension to multidimensional vector spaces of the well-known scalar fractional bilinear transformations. Projective matrix transformations have been found to map the scattering matrix, the impedance matrix, and the admittance matrix of an n-port network embedded in a 2n-port supernetwork. The transfer-scattering matrix and the chain- or ABCD-matrix of a 2n-port network embedded in a 4n-port supernetwork, are also mapped in a similar manner by matrix transformations of the same type. A fundamental application of this new transformation is the generalization of the concept of image-parameters known for 2-port networks to that of image-matrices for 2n-port networks. This generalization leads to a rigorous normal-mode analysis of wave-propagation on image-matched chains of cascaded 2n-port networks.

## 1. INTRODUCTION

A new and relatively unknown matrix transformation has been found to play a dominant role in the theory of microwave networks composed of cascaded multiport elements. The new "fractional bilinear matrix transformation" or "projective matrix transformation" was first discovered in the form of a multidimensional mapping of a scattering matrix, in the context of an investigation of new types of error modeling and calibration methods for automated network analyzers.[1,2] The extent of its relevance and the dominance of its role in the theory of microwave networks, composed of multiport elements was, however, not immediately recognized.

The ability of this type of matrix transform to map impedance and admittance matrices was subsequently discovered, and led to an extension of the well-known concept of image-impedance or image-admittance to 2n-port networks.[3]

The concept of image-matched, cascaded chains of 2-port networks was then extended to that of chains of image-matched 2n-port networks, and a very general analysis of the normal wave-modes propagating on such chains was made possible.

Quite recently, the ability of the fractional bilinear matrix transform to map the transfer-scattering matrix and the chain- or ABCD-matrix of a 2n-port network, embedded in a 4n-port supernetwork, was discovered.

One interesting aspect of these latest results is that the parameter-matrices to be used as representations of the embedding supernetwork, in the mapping of a T- or ABCD-matrix, are orthogonal transformations of the matrices to be used in mapping the corresponding S-matrix or Z- and Y-matrices, respectively. The required orthogonal matrices are block-permutation matrices.

It has also been discovered that the renormalization of the scattering matrix of a multiport network, with respect to a new and different set of complex, external port-impedances may be considered equivalent to embedding the network in an array of infinitesimally short, mismatched line junctions (a "cluster of junctions").[4] Here again, the projective matrix transform provides a general description of the scattering parameter transformation that performs the renormalization.

In view of the dominant role of the fractional bilinear matrix transform in microwave network theory, a partial investigation of its mapping properties has been conducted. Aspects of fundamental interest are the invariance and conservation properties of the transformation and their correlation to the response of the networks to be considered.[5]

Many more applications of the new matrix transformation are expected to follow from its recognition as a conspicuous, common thread among seemingly unrelated developments.

## 2. THE WELL-KNOWN SCALAR FRACTIONAL BILINEAR TRANSFORMATION

Scalar fractional bilinear transformations, defined as single-valued functions of a complex variable, are well-known to microwave engineers. In the theory of calibrated measurements of a complex reflection coefficient $\Gamma$, this type of transformation is written in the form[6]:

$$\Gamma_M = \frac{A \cdot \Gamma_X + B}{C \cdot \Gamma_X + 1} \tag{1}$$

In this context, the transformation (1) represents the relation between the true complex reflection coefficient $\Gamma_X$ of a single-port network X and the "uncalibrated" reflection coefficient reading $\Gamma_M$, observed in an imperfect 4-port reflectometer (Figure 1). This measurement system is composed of two cascaded directional couplers and a vector-voltmeter, connected to their side-arms.

Assuming linearity of the vector-voltmeter readings with respect to the complex ratio of the side-arm signals $\Gamma_M$, the given transformation (1) represents the effects of any mismatch, finite directivity or imperfect mutual tracking of the two directional couplers, and of any magnitude-ratio and phase errors in the vector-voltmeter.

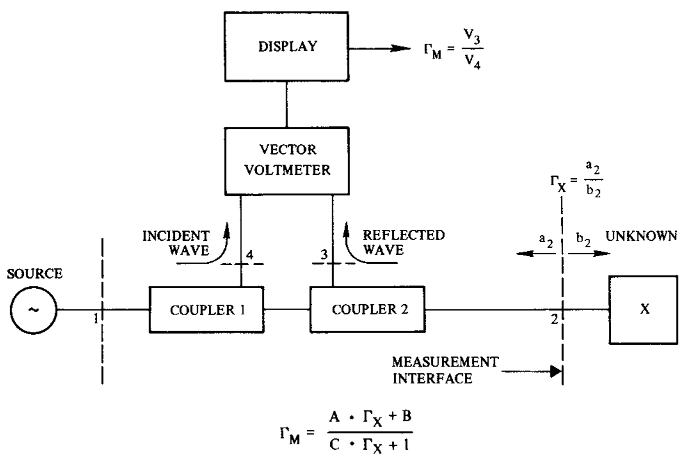


**Figure 1. Four-port reflectometer for the measurement of complex reflection coefficient $\Gamma_X$. $\Gamma_M$ is the observed complex vector ratio of the side-arm voltages.**

In the analysis of impedance transformers and matching networks, the scalar fractional bilinear transformation is often written in the form:

$$\Gamma_I = \frac{S_{11} - DET(S) \cdot \Gamma_L}{1 - S_{22} \cdot \Gamma_L} \tag{2}$$

where, $DET(S) = S_{11} S_{22} - S_{12} S_{21}$

In this context, the transformation, represented by equation (2) expresses the complex reflection coefficient $\Gamma_I$, that appears at the input port 1 of a given 2-port network N, when the output port 2 is terminated with a single-port load-network with reflection coefficient $\Gamma_L$ (Figure 2).

The scattering parameters $S_{ij}$, appearing in equation (2), characterize the 2-port network N that physically performs the transformation

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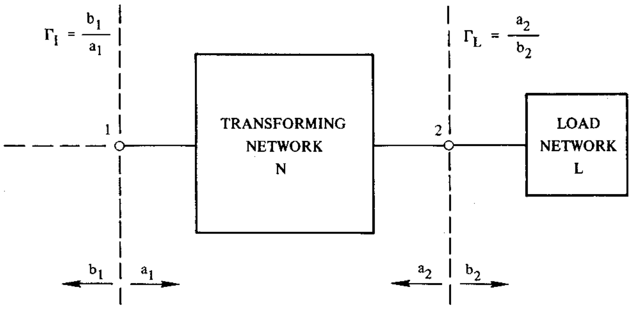
of the load-reflection $\Gamma_L$, connected at its output port, to the input reflection $\Gamma_I$, appearing at its input port. A transformation of this type represents the basis of the well-known Smith chart, in which case the transforming network N is a simple segment of uniform, lossless transmission line.




$$\begin{vmatrix} b_1 \\ a_1 \end{vmatrix} = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \cdot \begin{vmatrix} a_2 \\ b_2 \end{vmatrix} = \frac{1}{S_{21}} \begin{vmatrix} -\mathrm{DET}(S) & S_{11} \\ -S_{22} & 1 \end{vmatrix} \cdot \begin{vmatrix} a_2 \\ b_2 \end{vmatrix}$$

$$\Gamma_I = \frac{T_{11} \cdot \Gamma_L + T_{12}}{T_{21} \cdot \Gamma_L + T_{22}} = \frac{S_{11} - \mathrm{DET}(S) \cdot \Gamma_L}{1 - S_{22} \cdot \Gamma_L}$$

**Figure 2. Transformation of a complex reflection coefficient through a 2-port network.**

The scalar fractional bilinear transformation, exemplified in equations (1) and (2), is known to possess at least two remarkable properties. First, it maps circles in the complex $\Gamma_X$-plane to corresponding circles in the complex $\Gamma_M$ plane. Second, the Cross Ratio of four arbitrarily chosen complex values in the $\Gamma_X$-plane, is equal to the cross ratio of the corresponding points in the $\Gamma_M$ plane (conservation of the cross ratio):[7,8]

$$\frac{\Gamma_{M1} - \Gamma_{M2}}{\Gamma_{M1} - \Gamma_{M4}} \Big/ \frac{\Gamma_{M3} - \Gamma_{M2}}{\Gamma_{M3} - \Gamma_{M4}} = \frac{\Gamma_{X1} - \Gamma_{X2}}{\Gamma_{X1} - \Gamma_{X4}} \Big/ \frac{\Gamma_{X3} - \Gamma_{X2}}{\Gamma_{X3} - \Gamma_{X4}} \tag{3}$$

## 3. THE UBIQUITOUS FRACTIONAL BILINEAR MATRIX TRANSFORMATION

Recently[1,2,5] a multidimensional complex fractional bilinear matrix transformation was introduced in the form:

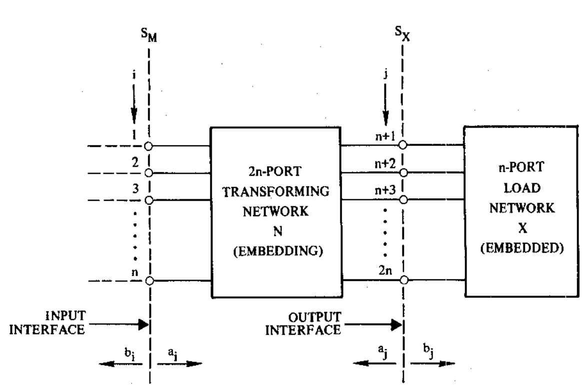
$$S_M = \left(T_1 \cdot S_X + T_2\right)\left(T_3 \cdot S_X + T_4\right)^{-1} \tag{4}$$

where the matrices $S_X$, $S_M$ and $T_i$ (i = 1, 2, 3, 4), are all complex n x n square matrices. The transformation of equation (4) was proved to represent the mapping of the complex n x n scattering matrix $S_X$, of an n-port load-network X, to the corresponding transformed input scattering matrix $S_M$, seen at the input of a 2n-port supernetwork N (Figure 3).

The n x n complex scattering matrix $S_M$ appears at the ports 1, 2, ..., n that constitute the input interface of the embedding supernetwork N, when the load-network X is connected to the remaining ports n + 1, n + 2, ..., 2n, at the output interface of the embedding supernetwork N. The supernetwork N is represented in equation (4) by its complex 2n x 2n transfer-scattering matrix:

$$T = \begin{vmatrix} T_1 & T_2 \\ T_3 & T_4 \end{vmatrix} \tag{5}$$

where, the $T_i$'s (i = 1, 2, 3, 4) are the four n x n blocks or "quadrants" of the matrix T. An interesting aspect of the transformation of equation (4) is that the individual n x n blocks or quadrants $T_i$ (i = 1, 2, 3, 4), of the transfer scattering matrix T, separately appear as matrix-parameters of the fractional bilinear matrix transformation linking $S_M$ to $S_X$.




$$\begin{vmatrix} b_i \\ a_i \end{vmatrix} = \begin{vmatrix} T_1 & T_2 \\ T_3 & T_4 \end{vmatrix} \cdot \begin{vmatrix} a_j \\ b_j \end{vmatrix} \qquad \begin{vmatrix} a_{n+1} \\ \vdots \\ a_{2n} \end{vmatrix} = S_X \cdot \begin{vmatrix} b_{n+1} \\ \vdots \\ b_{2n} \end{vmatrix} \qquad \begin{vmatrix} b_1 \\ \vdots \\ b_n \end{vmatrix} = S_M \cdot \begin{vmatrix} a_1 \\ \vdots \\ a_n \end{vmatrix}$$

$$S_M = (T_1 \cdot S_X + T_2)\,(T_3 \cdot S_X + T_4)^{-1}$$

**Figure 3. Transformation of an n x n complex scattering matrix $S_X$ through a 2n-port transforming network. The transforming network N may be thought of as embedding or encircling the load-network X totally and is characterized by a 2n x 2n transfer-scattering matrix T with n x n quadrants $T_1$, . . . , $T_4$.**

In references 1 and 2, a method was developed for computing the four quadrants $T_i$ of the matrix T from pairs of corresponding $S_{Xi}$, $S_{Mi}$ matrices. This method provides a way for indirectly characterizing the embedding supernetwork N from an external analysis of its transformation properties. The method combines a generalized gaussian condensation,[9] applied to a set of $4n^2$ homogeneous, scalar linear equation, and an explicit solution of various linear matrix equations.[10]

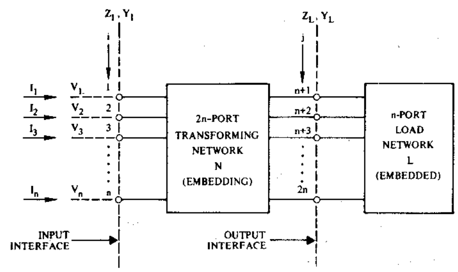
Subsequently,[3] new fractional bilinear matrix transformations, exemplified by equation (4), were found to describe the mapping of the Z-matrix $Z_L$ and of the Y-matrix $Y_L$ to the corresponding input-interface Z- and Y-matrices $Z_I$ and $Y_I$, seen at the input interface of 2n-port supernetwork N (Figure 4). These new transforms may be written in the form:

$$Z_I = \left(A \cdot Z_L + B\right)\left(C \cdot Z_L + D\right)^{-1} \tag{6}$$

$$Y_I = \left(D \cdot Y_L + C\right)\left(B \cdot Y_L + A\right)^{-1} \tag{7}$$

where A, B, C, D are the n x n quadrants of the 2n x 2n chain-matrix K of the embedding supernetwork N, as defined by:

$$\begin{vmatrix} V_I \\ I_I \end{vmatrix} = K \cdot \begin{vmatrix} V_J \\ -I_J \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \cdot \begin{vmatrix} V_J \\ -I_J \end{vmatrix} \tag{8}$$




$$\begin{vmatrix} V_i \\ I_i \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \cdot \begin{vmatrix} V_j \\ -I_j \end{vmatrix} \qquad \begin{vmatrix} V_1 \\ \vdots \\ V_n \end{vmatrix} = Z_I \cdot \begin{vmatrix} I_1 \\ \vdots \\ I_n \end{vmatrix} \qquad \begin{vmatrix} V_{n+1} \\ \vdots \\ V_{2n} \end{vmatrix} = Z_L \cdot \begin{vmatrix} -I_{n+1} \\ \vdots \\ -I_{2n} \end{vmatrix} \qquad i = 1, 2, \ldots, n$$

$$\begin{vmatrix} I_1 \\ \vdots \\ I_n \end{vmatrix} = Y_I \cdot \begin{vmatrix} V_1 \\ \vdots \\ V_n \end{vmatrix} \qquad \begin{vmatrix} -I_{n+1} \\ \vdots \\ -I_{2n} \end{vmatrix} = Y_L \cdot \begin{vmatrix} V_{n+1} \\ \vdots \\ V_{2n} \end{vmatrix} \qquad j = n+1, n+2, \ldots, 2n$$

$$Z_I = (A \cdot Z_L + B)\,(C \cdot Z_L + D)^{-1} \qquad Y_I = (D \cdot Y_L + C)\,(B \cdot Y_L + A)^{-1}$$

**Figure 4. Transformation of an n x n complex impedance matrix $Z_L$ or of an n x n complex admittance matrix $Y_L$ through a 2n-port transforming network N. The network N may be thought of as embedding or encircling the load-network L totally and is characterized by a 2n x 2n ABCD or chain-matrix K with n x n quadrants A, B, C, D.**

Two more, as yet unpublished, results have now been obtained. These new results describe the embedding of a 2n-port load-network X, by a 4n-port supernetwork N (Figure 5), in terms of the mapping of its transfer-scattering matrix $T_X$, or its chain-matrix $K_X$.

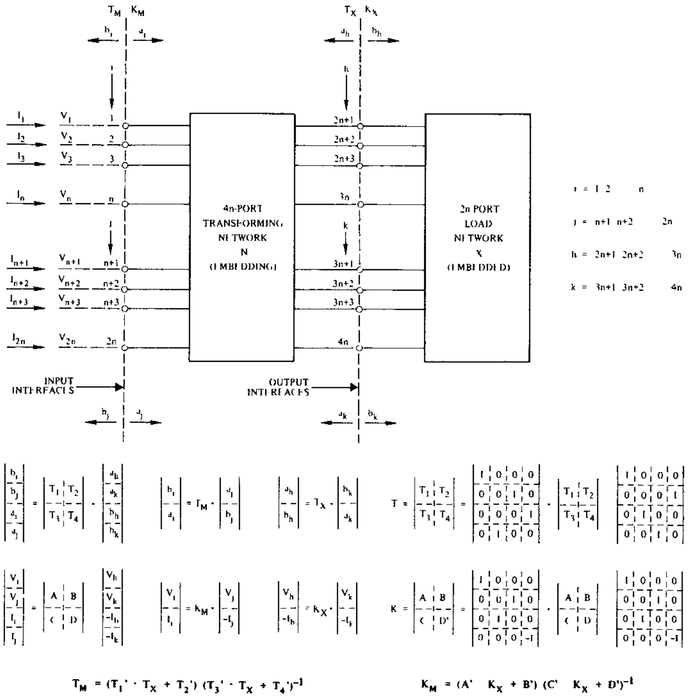



**Figure 5. Transformation of a 2n x 2n complex transfer scattering matrix $T_X$ or of a 2n x 2n complex chain matrix $K_X$ through a 4n-port transforming network N. The transforming network N may be thought of as embedding or encircling the load-network totally and is characterized by the modified 4n x 4n matrix T' or the modified 4n x 4n matrix K'. The matrices T' and K' are orthogonal transformations of the T-matrix T and of the chain-matrix K, respectively.**

In the first of these new results, the embedded 2n-port network X is represented by its transfer-scattering matrix $T_X$, and the 4n-port embedding supernetwork N is represented by a modified 4n x 4n T-matrix T', with quadrants $T_i'$ (i = 1, 2, 3, 4). This first result may be written in the form:

$$T_M = \left(T_1' \cdot T_X + T_2'\right)\left(T_3' \cdot T_X + T_4'\right)^{-1} \qquad (9)$$

where $T_M$ represents the mapping of the T-matrix $T_X$ from the two output interfaces of the supernetwork N (ports h: $2n + 1 \leqslant h \leqslant 3n$; and ports k: $3n + 1 \leqslant k \leqslant 4n$) to the two input interfaces of N (ports i: $1 \leqslant i \leqslant n$; and ports j: $n + 1 \leqslant j \leqslant 2n$). In equation (9) the quadrants $T_i'$, of the modified 4n x 4n T-matrix T' of the supernetwork N, are defined by:

$$T' = \left|\begin{array}{c|c} T_1' & T_2' \\ \hline T_3' & T_4' \end{array}\right| = P_1^T \cdot T \cdot P_1 = P_1^T \cdot \left|\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right| \cdot P_1 \qquad (10)$$

where $P_1$ is the orthogonal block-permutation matrix:

$$P_1 = \left|\begin{array}{c|c|c|c} I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \\ \hline 0 & I & 0 & 0 \end{array}\right| \qquad (11)$$

Similarly, in the second new result, the embedded 2n-port network X is represented by its 2n x 2n chain-matrix $K_X$ and the 4n-port embedding supernetwork N is represented by a modified 4n x 4n chain-matrix K', with quadrants A', B', C', D'. This second result may be written in the form:

$$K_M = \left(A' \; K_X + B'\right)\left(C' \cdot K_X + D'\right)^{-1} \qquad (12)$$

where $K_M$ represents the mapping of the chain-matrix $K_X$ of X from the two output interfaces of the supernetwork N (ports h, k) to the two input interfaces of N (ports i, j).

In equation (12) the quadrants A', B', C', D' of the modified 4n x 4n chain-matrix K' are defined by:

$$K' = \left|\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right| = P_2^T \cdot K \cdot P_2 = P^T \cdot \left|\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right| \cdot P_2 \qquad (13)$$

where $P_2$ is the orthogonal and autoinverse block-permutation matrix:

$$P = \left|\begin{array}{c|c|c|c} I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & -I \end{array}\right| \qquad (14)$$

The transformations of equations (6), (7) and (12) are unconditionally true in the embedding situations of Figures 4 and 5, respectively. The transformations (4) and (9) are true, under the condition that the bases of normalization for the matrices $S_X$ and $S_M$ or $T_X$ and $T_M$ are "compatible" with those of the matrices T or T' of the embedding supernetwork N at its output interface (for $S_X$ and $T_X$) and its input interface (for $S_M$ and $T_M$), respectively. This "compatibility" of the normalization bases means equality of the normalizing complex impedances of corresponding ports, in the case of voltage-waves (traveling waves) and, conversely, mutual complex conjugation of these normalizing impedances in the case of power-waves.

## 10. REFERENCES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